Equations: System of Linear Equations

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A set of two linear equations E_1 and E_2 in the same two variables (e.g., x and y) is called a ***system of linear equations***. The equations are called *simultaneous equations*. A solution to such a system is a pair of numbers (p, q) that is a solution to *both* equations. This is called the *simultaneous solution* to the equations.

It is important to see how this simultaneous solution to both equations is related to the solution to each equation.

Recall that the solution to one linear equation in two variables x and y is, in general, all the points lying on a line in the x-y plane.

Suppose now that two such equations E_1 and E_2 are given:

- The solution to just equation E_1 is all the points (x,y) lying along some line L_1 in the plane.
- The solution to just equation E_2 is all the points (x, y) lying along some line L_2 in the plane.
- It follows that the solution to both equations E_1 and E_2 is the point (p, q) that lies at the intersection of lines L_1 and L_2.

Stated again, you can say:

The simultaneous solution to two linear equations in two variables is the point at the intersection of the two lines that are the graphs of the equations.

Here is an example of a system of two simultaneous equations in two variables:

$$3x + 8y = 28 \qquad 8x - 5y = 22$$

The simultaneous solution to these two equations is the pair of numbers $x = 4$ and $y = 2$. This pair corresponds to the point in the plane $(x, y) = (4, 2)$.

This is a simultaneous solution because substituting these values for x and y leads to a true numerical equation in both equations:

$$3x + 8y = 28 \qquad\qquad 8x - 5y = 22$$
$$3 \cdot 4 + 8 \cdot 2 = 28 \qquad\qquad 8 \cdot 4 - 5 \cdot 2 = 22$$

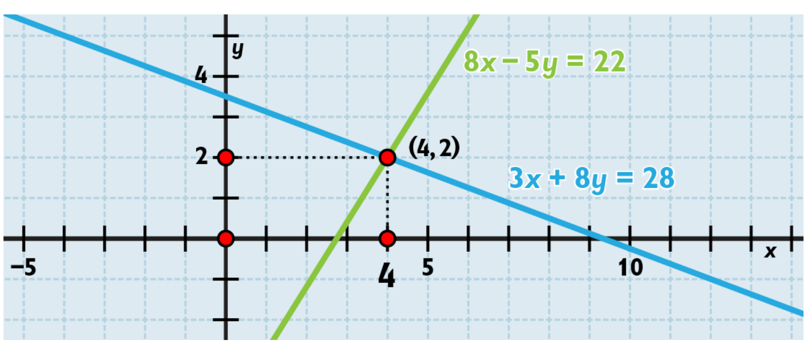
You can find this solution either *graphically* or *algebraically*.

Graphing the Solution

To find the solution graphically, you start by graphing each line.

Here are the graphs of the two linear equations:

$$3x + 8y = 28 \qquad 8x - 5y = 22$$



For each line graph, all the points (x, y) on that line are solutions to the equation of that graph. Therefore, the simultaneous solution to both solutions is the intersection (4, 2) of the two lines.

Finding the Solution Algebraically

You just saw that when the two equations in a system of linear equations in two variables are graphed, the solution is easy to find: it is the point (p, q) at the intersection of the two graphs. It is useful to be able to find a solution without having to graph the equations.

The basic principle for solving a system of two equations algebraically is simple: solve one equation for one variable in terms of the other. Then substitute this solution expression in the other equation. This is called the *substitution method*.

Substitution Method

Consider again these two simultaneous equations in two variables:

$$3x + 8y = 28 \qquad 8x - 5y = 22$$

You can apply the substitution method by solving the first equation for y in terms of x and then substituting this value into the second equation:

$3x + 8y = 28$	first given equation
$y = 3.5 - 0.375x$	solve for y in terms of x: addition property of equality and multiplication property of equality

$8x - 5y = 22$	second given equation
$8x - 5(3.5 - 0.375x) = 22$	substitute the value for y into the second equation
$8x - 17.5 + 1.875x = 22$	distributive property
$9.875x = 39.5$	combine like terms and distributive property
$x = 4$	multiplication property of equality

$3(4) + 8y = 28$	substitute $x = 4$ into the first equation
$y = 2$	solve for y

The result is the same solution found graphically: $x = 4$, $y = 2$.

Elimination Method

The substitution method is very general and always works. But the arithmetic can be messy! By inspecting the system carefully, you can sometimes make the arithmetic easier by adding the equations and eliminating a variable. This is called the *addition/subtraction method* or the *elimination method.*

In this method, you solve a system of two linear equations E_1 and E_2 in two unknowns by adding (or subtracting) the two equations to form a single linear equation E_3 that you can use to replace E_1 or E_2. The justification for the method is that any simultaneous solution of E_1 and E_2 is also a solution to E_3.

Suppose a point (p, q) is a solution of both equations E_1 and E_2. Then (p, q) is also a solution to E_3, since equals added to equals result in an equal.

Since $x = p$ and $y = q$ is the desired simultaneous solution to the two equations, this means that equation E_3 is a valid equation to use to find a solution to the system. Often, equation E_3 is an equation in just one variable that you can solve immediately for that variable.

Look again at this system of two simultaneous equations in two variables:

$$3x + 8y = 28 \qquad 8x - 5y = 22$$

Suppose you multiply the first equation by 5 and the second equation by 8. This does not change the solutions.

$15x + 40y = 140$	multiply first equation by 5
$64x - 40y = 176$	multiply second equation by 8

Now, add the previous two equations.

$79x = 316$	add
$x = 4$	divide both sides by 79

Substitute $x = 4$ into either the first or second equation to solve for y. The result is the full solution to the system: $x = 4$ and $y = 2$.

Sometimes the elimination method can work very quickly. Consider these two linear equations in two unknowns:

$2x - 3y = -5$
$4x + 3y = 17$

If you *add* these two equations, you get $6x = 12$. You have eliminated the variable y. The solution to this linear equation in one variable is easy to find: $x = 2$.

From this you can find the solution $y = 3$ by substituting 2 into either equation. Thus, the full solution is $x = 2$ and $y = 3$.

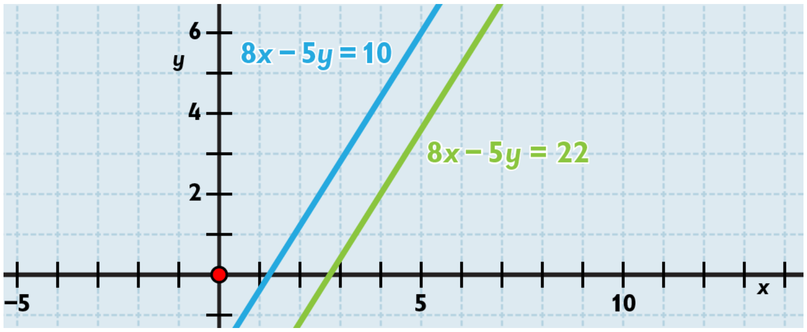
A System With No Solutions

Although most systems of equations will have one unique solution, it is possible for the equations to have no solutions.

This is an example of a system of two equations in two variables with no solution:

$$8x + 5y = 22 \qquad 8x - 5y = 10$$

The graphs show why there is no solution:



The line graphs of the two equations are parallel. This makes it clear why there is no solution: parallel lines do not intersect!

Actually, it should be clear from the equations themselves why there can be no solution. Whatever numerical values x and y have, $8x - 5y$ cannot equal both 22 and 10 at the same time.

A pair of equations such as $8x - 5y = 22$ and $8x - 5y = 10$ is called *inconsistent*. The one equation makes a statement about the relationship of x and y that is inconsistent with the other equation. If these two equations had been formulated to model a problem situation with unknown quantities x and y, the inconsistent equations show that no quantities x and y have the desired properties.

A System Where Every Number Is a Solution

It is also possible for a system of two equations to have an infinite number of solutions. Here is an example:

$$8x - 5y = 22 \qquad 5y = 8x - 22$$

If you graph these two equations, you would see that their graphs are identical. This shows why there are an infinite number of solutions. In particular, every point (x, y) on the graph is a solution.

If these two equations had been formulated to model a problem situation with unknown quantities x and y, it is clear that they each describe the same condition. That is, they are not *independent.* To narrow the solution down to just one pair of numbers $x = p$ and $y = q$, you need another independent equation.

You can see from the equations themselves why the graphs are identical. One is just a re-arrangement of the terms of the other.