Proportional Relationships: Representations

What Is a Proportional Relationship?

Think about a cube: a cube has six faces. If you have a bag of cubes, there is a definite relationship between the total number of faces and the number of cubes in the bag: there are six times as many faces as cubes.

This very simple type of relationship is called a *proportional relationship*. In any collection of cubes, the number of faces is proportional to the number of cubes.

Constant Ratio, Constant Multiple, and Constant of Proportionality

You can express the proportional relationship between the number of cubes in the bag (n) and the number of faces of those cubes (F) as follows:

1. The ratio of F and n is constant: Fn = 6
2. F is a constant multiple of n: $F = 6n$

These two statements express the same thing. The first one states that two quantities (F and n) have a *constant ratio*. The second one states that one quantity (F) is a *constant multiple* of the other quantity (n). In each case, the number 6 is called the *constant of proportionality*.

The relationship between faces and number of cubes can also be looked at with the roles of F and n reversed. The following statements are two equivalent ways of expressing this relationship:

3. The ratio of n and F is constant: nF=16
4. n is a constant multiple of F: n = (16)F

These two statements express the same thing. As with statements (1) and (2), statement (3) says that two quantities have a constant ratio. Statement (4) says that one quantity is a constant multiple of the other quantity. However, in these statements the constant of proportionality is the number 16.

Notice that the two constants of proportionality, 6 and 16, are *reciprocals*.

Constant of Proportionality and "Per"

Consider the constant multiple formula describing the relationship between faces and number of cubes:

$F = 6n$

This relationship is often described using the special word "per," along with the constant of proportionality, 6. You would say:

There are 6 faces per cube.

This means that for every cube there are 6 faces. Thus, for n cubes there are $6n$ faces. The formula $F = 6n$ tells you how to find the number of faces (F) for any number of cubes (n).

Constant of Proportionality and Rate

In the formula $F = 6n$, the constant of proportionality, 6, can also be called a *rate*.

The rate for the formula $F = 6n$ tells you that there are "6 faces per cube."

A rate is also sometimes called a *unit rate*. The word "unit" means "1," and a unit rate of 6 faces per cube simply means that for every 1 cube there are 6 faces.

You may be familiar with the word "rate" from everyday examples such as "rate of speed." For example, if a bicycle is traveling at a speed of 6 m per second, you can relate the distance d traveled (in meters) to the time t of travel (in seconds):

$d = 6t$

In this situation the distance traveled (d) is proportional to the time of travel (t). The constant of proportionality, 6, is a rate that gives the rate of speed of the bicycle: 6 m per sec.

Units

In a formula such as $d = 6t$, each of the three quantities is associated with a unit of measure:

d Distance traveled; the unit is "meters"

t Time of travel; the unit is "seconds"

6 Constant rate of speed; the unit is "meters per second"

The unit for the constant of proportionality (meters per second) explicitly relates the unit "meters" and the unit "seconds" using the special term "per."

To better understand the meaning of the unit "meters per second," you need to look at how a rate is found using division.

The Role of Division in a Rate

Consider again the bicycle situation described by the formula $d = 6t$.

In this situation, 6 is the rate of speed: 6 m per sec. You might wonder how this rate was determined. One answer is that someone may have measured a particular distance *d,* tracked the time *t* that it took for the bicycle to travel that distance, and then divided one value by the other. For example, suppose these are the measurements:

The bicycle covered the distance d = 60 m in the time t = 10 sec.

Then the rate of speed can be found by using division:

The unit meterssecond on the right-hand side is written in words as “meters per second.” So, the unit “meters per second” means “meters divided by seconds.”

The structure of the formula $d = 6t$ can be seen clearly if the units of the three quantities are shown in this way:

$d = 6 \bullet t$

meters = meterssecond • seconds

In this formula, the two occurrences of the unit “seconds” cancel each other out, as in a fraction. So, the units of the formula as a whole are simple:

meters = meters

Every formula must be consistent in its use of units in just this way.

Representing Proportional Relationships

Tables

A ratio table shows the equal ratios in a proportional relationship. Here is a ratio table for the cube example:

Number of Cubes (n)	0	1	2	3	4	5	…	n
Number of Faces (F)	0	6	12	18	24	30	…	$6n$

About ratio tables:

- Every ratio table has (0, 0) as a pair of corresponding values.
- Each column in a ratio table has two numbers. The number in the bottom row is always the result of multiplying the number in the top row by a constant. In the table, this constant is 6.
- You can find new pairs by multiplying each number in any existing pair of values (except 0) by the same number.
- You can also find new pairs by adding (or subtracting) other pairs in the table. For example, for pairs (n, F) in the table, adding the pair (2, 12) to the pair (3, 18) gives the pair (5, 30).
- You can use these strategies to find any unknown quantity, such as 30 faces for 5 cubes in the table.

Graphs

The graph of a proportional relationship is a straight line through the origin (0, 0). The *slope* is equal to the constant of proportionality. The following graphs show the proportional relationship between the number of cubes and the number of faces. In the first graph, the slope is 6. In the second graph, the slope is 16.

Each point corresponds to a column in the table (from the previous section):

Number of Cubes (n)	0	1	2	3	4	5	…	n
Number of Faces (F)	0	6	12	18	24	30	…	$6n$

Examples of Proportional Relationships

The following are everyday examples of different types of proportional relationships. Note that the examples are based on the general formula for a proportional relationship:

$p = kq$

In this formula, p and q are the related quantities and k is the constant of proportionality.

1. The quantities p and q both represent amounts of things.

a. Chairs/tables: k is the number of chairs at each table. Different types of tables have different values of k, but $k = 4$ for a common type of table. This situation can be represented with the following formula:

$c = 4t$ where c = chairs and t = tables

The constant of proportionality is 4, or "4 chairs per table."

b. Wheels/cars: k is the number of wheels on a car. Different types of cars have different values of k, but $k = 4$ for most cars. This situation can be represented with the following formula:

$w = 4c$ where w = wheels and c = cars

The constant of proportionality is 4, or "4 wheels per car."

c. Students/teachers: k is the student/teacher ratio. Suppose there are 20 students in a class. This situation can be represented with the following formula:

$s = 20t$ where s = students and t = teachers

The constant of proportionality is "20 students per teacher."

2. The quantities p and q are both lengths, and k expresses a size relationship, called the *scale factor* or *similarity ratio*.

a. Similar triangles: Corresponding sides are proportional; k is the ratio of similarity.

b. Maps or floor plans: k is the scale factor.

c. Enlargement/reduction: k is a photocopier setting (often given as a percent). For example, here is the formula for enlarging by a factor of 2:

$c = 2d$ where d = a length in the original and c = the corresponding length in the copy

(A length c in the copy is 2 times the corresponding length d in the original.)

3. The quantities p and q are both lengths, and k expresses a shape relationship.

a. Steepness of ramps, stairs, roads, or roofs: k is the slope, and it is expressed in at least two different ways (e.g., a 6% grade on a road; a 1 to 5 slope of a roof). For example, here is the formula for a road with a grade of 6%:

$v = 0.06\,h$ where v = vertical rise and h = horizontal run

The vertical rise v over any length of road is proportional to the horizontal run h. The constant of proportionality is 0.06 units of rise per unit of run.

b. Slope: In a simple case, where vertical and horizontal scales are the same, the mathematical slope of a graph is the same as the geometric slope. In contrast, if vertical and horizontal scales are not the same, the mathematical slope is different from the geometric slope.

4. The quantities p and q are both monetary values.

a. Cost/sales tax: k is the sales tax rate, usually expressed as a percent.

b. Cost/discount: k is the percent of the discount amount in a sale (e.g., every item 20% off). For example,

$s = 0.80p$ where s = the sale price and p = the original price

The sale price s is 80% of the original price p.

5. The quantities p and q are any quantities of the same type, and k is the *conversion factor*.

a. Centimeter/inch:

$c = 2.54i$ where i is a length measured in inches and c is the same length measured in centimeters

The constant of proportionality, 2.54, has the unit "centimeters per inch," meaning that there are 2.54 cm in 1 in.

Notice that the value of c will always be 2.54 times the corresponding value of i, since centimeters are shorter than inches. A stick that is 2 in. long is 5.08 cm long.

b. Minute/hour:

$m = 60h$ where h is a time interval in hours and m is the same time interval in minutes

There are always 60 min in 1 hr. The constant of proportionality, 60, has the unit "minutes per hour."